Indian Statistical Institute
B. Math. I Year
Semestral Examination 2008-2009

Date: 04-05-2009 Algebra II Instructor: J. Biswas

Attempt all questions. Total Marks 50

1)Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z)^t = (x + z, x - y + z, 2y)^t$.

a)What is the matrix of T with respect to the standard basis on $\mathbb{R}^3$? What is the matrix of T with respect to the basis $\{(1, 1, 0)^t, (1, 0, 1)^t, (0, 1, 1)^t\}$? Verify the change of basis formula for T with respect to these two bases.

b)What is the rank and nullity of T?

c)What are the eigenvalues of T? Is T diagonalizable? (5+2+3)

2)Let V be the real vector space of 2×2 matrices over $\mathbb{R}$ and define $\langle A, B\rangle = det(A + B) - det(A) - det(B)$.

a)Prove that $<,>$ defined above is a symmetric, bilinear form on V.

b)Compute the matrix of this form with respect to the standard basis $\{e_{ij}\}$ and determine the signature of this form.

c) Determine the signature of the above form restricted to the subspace of V consisting of matrices of trace zero. (2+4+4)

3)a)State and prove the analogue of Sylvester's law for symmetric forms over complex vector spaces.

b)Prove that a Hermitian form over a complex vector space V has an orthonormal basis if and only if it is positive definite.

c)Show that the only complex matrix which is positive definite, hermitian and unitary is the identity matrix. (6+6+6)

4a)State the spectral theorem for hermitian operators on a hermitian vector space and use it to prove that a positive definite real symmetric $n \times n$ matrix P has the form $P = AA^t$ for some $n \times n$ real matrix A.

b)Let A be a $m \times n$ matrix of rank r over any field. Prove that some $r \times r$ minor of A is invertible and all $(r+1) \times (r+1)$ minors of A have determinant zero. (6+6)